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ON COALESCENCE OF FREQUENCIES AND CONICAL POINTS IN THE DISPERSION SPECTRA OF ELASTIC BODIES

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Abstract—In this paper we discuss a general procedure for determining the critical points of the dispersion spectrum at which there is a coalescence of frequencies, i.e. critical points which are roots of double multiplicity. We further show how the general behavior of the dispersion surface in the neighborhood of the critical points can be determined analytically. For the purpose of illustration, we consider (a) plane waves propagating in an infinite, elastic, isotropic plate, which corresponds to the case of a differential equation with constant coefficients, and (b) Floquet waves of the SH-type propagating in a layered, elastic medium, which corresponds to the case of a differential equation with periodic coefficients.

1. INTRODUCTION

In the study of dynamic behavior of linearly elastic bodies, dispersion spectra, i.e. relations between the frequency of free vibrations and the components of the vector wave-number, occupy a place of central importance. For homogeneous, isotropic bodies of infinite extent such spectra are single-valued, i.e. for every given value of the vector wave-number there corresponds a unique value of the frequency. By contrast, in bounded or inhomogeneous bodies (e.g. plates, shells, laminated composites, etc.) single-valuedness may no longer exist for certain selected values of the wavenumber, i.e. two or more distinct values of the frequency are determined for a given value of the wave-number. For certain choices of the system parameters it may even happen that two distinct neighboring values of the frequency may coalesce for special values of the wave-number, corresponding to what will be termed here as critical or *conical* points.

The present paper is devoted to a systematic study of such critical points in dispersion spectra of elastic bodies. The necessary and sufficient conditions are established, under which the frequency equation will have double roots. Next, the nature and properties of the dispersion spectrum in the vicinity of the critical points are analyzed. By using a two-dimensional wave-vector, it is shown that the dispersion spectrum in the neighborhood of the critical point is a conical surface. Thus the critical point itself may be called a conical point.

As a first simple illustration of the general development, the coalescence of cut-off frequencies of Rayleigh–Lamb waves in a homogeneous, isotropic, elastic plate of uniform thickness is considered. As a second illustration of the general theory, the dispersion spectrum of Floquet waves which are horizontally polarized and propagating in a periodically-layered composite of infinite extent is investigated. It is shown that in this case the conical points can exist only at the ends of the Brillouin zone, and their properties are discussed in detail.

It is expected that other examples could be presented to make use of the general theory. From topological considerations it can be shown that for a single-valued function defined over a domain which is periodic in the reciprocal lattice, there exist a number of critical points. Consequently, critical points of various types will exist in dispersion spectra of elastic bodies with periodic structure. However, their discussion will be deferred to a possible future study.

2. CONTACT POINTS AND THE NATURE OF THE DISPERSION SURFACE

In this section we first discuss the necessary and sufficient conditions for the frequency equation to have *non-simple* roots, i.e. multiplicity of order two. We then analyze the nature of the dispersion surface in the neighborhood of this non-simple zero of multiplicity two. The existence of the multiple roots of the frequency equation was apparently first noticed by Mindlin [1] in his study of the dispersion spectrum of plane waves in an infinite, isotropic plate, and he named this phenomenon "*coincidence* of thickness frequencies", that is, the coalescence of two distinct but neighboring frequencies of the normal modes of the plate. His study in the present context can be described as the study of contact points of order one of a frequency equation with one independent wave-number. Furthermore, it may appear that he restricted his analysis to *cut-off* frequencies only, that is, when the traveling waves change to attenuated waves, at infinite wavelength. Recently, in a much more general context, in dealing with differential equations with periodic coefficients, coalescence of frequencies has been noticed in the case of Floquet waves in periodic structures. When two independent wave-numbers are involved, the dispersion spectrum for Floquet waves is a surface, exhibiting *stopping* bands and *passing* bands and a periodicity of the reciprocal lattice distance. However, if the dispersion surface has a contact point, the stopping band disappears, effectively doubling the width of the Brillouin zone, and thus increasing the band width of the passing band.

In the case of Floquet waves these contact points are important features of the dispersion surface. This investigation is concerned with the analysis of the contact points and the nature of the dispersion surface in the neighborhood of these critical points. For the sake of completeness, we first consider the problem in its generality.

Analysis. Consider a spectral frequency equation

$$F(\Omega, \lambda, \kappa) = 0, \tag{1}$$

corresponding to a self-adjoint differential operator, where Ω is the non-dimensional frequency and the two independent variables λ, κ are the two non-dimensional wave-numbers for waves propagating in two mutually orthogonal directions. The frequency Ω is always real, but the two wave-numbers can be complex. We assume that in the neighborhood of a typical point P_0: $\{\Omega_0, \lambda_0, \kappa_0\}$, the function F together with its three first partial derivatives are continuous. If at this point

$$\begin{gathered} F(\Omega_0, \lambda_0, \kappa_0) = 0, \\ \left(\frac{\partial F}{\partial \Omega}\right)_0 = \left(\frac{\partial F}{\partial \lambda}\right)_0 = \left(\frac{\partial F}{\partial \kappa}\right)_0 = 0, \end{gathered} \tag{2}$$

then the point P_0 is a critical point of the spectral function F and is called a multiple root of order two. The contact of the surfaces at this point is of order one. Contact points of the spectral surface corresponding to order 1 (*odd*), are the critical points at which there is coalescence of frequencies of two different spectral surfaces. The exact location of such points in the $(\Omega, \lambda, \kappa)$-affine space depends on the simultaneous solution of the system of eqns (2), that is, at these analytical critical points the gradient vector of the dispersion surface vanishes identically.

We now determine the shape of the surface in the neighborhood of the critical point P_0, where the three first derivatives of the spectral function F are assumed to vanish simultaneously. Using Taylor series expansion with Lagrange remainder, and assuming that the three second derivatives do not vanish at point P_0, and that these derivatives together with the third derivatives are continuous near the point, we get

$$F(\Omega, \lambda, \kappa) = \frac{1}{2!} D^2 F(\Omega, \lambda, \kappa)|_{P_0} + \frac{1}{3!} D^3 F(\Omega*, \lambda*, \kappa*)|_{P*} = 0, \tag{3}$$

where

$$D^n \equiv \left[(\Omega - \Omega_0)\frac{\partial}{\partial \Omega} + (\lambda - \lambda_0)\frac{\partial}{\partial \lambda} + (\kappa - \kappa_0)\frac{\partial}{\partial \kappa}\right]^n, \tag{4}$$

and $P* \equiv \{(\Omega*, \lambda*, \kappa*)$: $\Omega_0 + t*(\Omega - \Omega_0)$, $\lambda_0 + t*(\lambda - \lambda_0)$, $\kappa_0 + t*(\kappa - \kappa_0)\}$, $0 < t* < 1$. We may now note that in the Taylor expansion, if we omit the third order terms and set the terms of second degree in $(\Omega - \Omega_0)$, $(\lambda - \lambda_0)$ and $(\kappa - \kappa_0)$ equal to zero, we obtain the equation of the cone

$$\left[(\Omega - \Omega_0)\frac{\partial}{\partial \Omega} + (\lambda - \lambda_0)\frac{\partial}{\partial \lambda} + (\kappa - \kappa_0)\frac{\partial}{\partial \kappa}\right]^2 F(\Omega, \lambda, \kappa)|_{P_0} = 0. \tag{5}$$

It can be shown that this equation represents the locus of the tangents to all curves on the surface passing through the critical point P_0. The point P_0 is therefore called a *conical* point[2].

Consider now a plane $\kappa = \kappa_0$ passing at the point P_0 through the surface of the cone. The equation of the curve of intersection, in the neighborhood of the point P_0, is

$$\frac{1}{2!}\left[(\Omega-\Omega_0)\frac{\partial}{\partial\Omega}+(\lambda-\lambda_0)\frac{\partial}{\partial\lambda}\right]^2 F(\Omega,\lambda,\kappa_0)|_{P_0} + \frac{1}{3!}\left[(\Omega-\Omega_0)\frac{\partial}{\partial\Omega}+(\lambda-\lambda_0)\frac{\partial}{\partial\lambda}\right]^3 F(\Omega*,\lambda*,\kappa_0)|_{P*}=0. \quad (6)$$

To obtain the equation of the tangent to the curve of intersection, we assume that $F_{\Omega_0\Omega_0}\neq 0$, and set

$$(\Omega-\Omega_0)=\alpha(\lambda-\lambda_0), \quad (7)$$

in eqn (6). Then, on dividing by $(\lambda-\lambda_0)^2$, the resulting equation takes the form

$$\alpha^2\frac{\partial^2 F}{\partial\Omega_0{}^2}+2\alpha\frac{\partial^2 F}{\partial\Omega_0\partial\lambda_0}+\frac{\partial^2 F}{\partial\lambda_0{}^2}+(\lambda-\lambda_0)Q(\lambda-\lambda_0;\alpha)=0, \quad (8)$$

where $(\partial^2 F/\partial\Omega_0{}^2)\equiv(\partial^2 F/\partial\Omega^2)_{P_0}\equiv F_{\Omega_0\Omega_0}$, etc., and $Q(\lambda-\lambda_0;\alpha)$ is a function assumed to remain finite as λ approaches λ_0. Equation (8) can easily be rewritten in the form

$$(\alpha-\alpha^+)(\alpha-\alpha^-)F_{\Omega_0\Omega_0}+(\lambda-\lambda_0)Q=0, \quad (9)$$

where

$$\left.\begin{matrix}\alpha^+\\ \alpha^-\end{matrix}\right. = [-F_{\Omega_0\lambda_0}\pm(F^2_{\Omega_0\lambda_0}-F_{\Omega_0\Omega_0}F_{\lambda_0\lambda_0})^{1/2}]F^{-1}_{\Omega_0\Omega_0}. \quad (10)$$

When the discriminant is real, the two distinct roots (α^+,α^-) are both real. By assumption the function Q is bounded, and therefore as λ approaches λ_0, the two distinct roots of the equation approach the values α^+ and α^-, respectively. Therefore it follows that eqn (8) has roots of the form

$$\Omega=\Omega_0+(\lambda-\lambda_0)(\alpha^++\epsilon), \quad (11)$$

where ϵ approaches zero as λ approaches λ_0. Hence, there is one branch of the spectral curve passing through the critical point P_0 and lying in the plane $\kappa=\kappa_0$, which is tangent to the straight line

$$\Omega=\Omega_0+\alpha^+(\lambda-\lambda_0), \quad (12)_1$$

where α^+ is the slope of the line and given by eqn $(10)_1$. It can similarly be shown that the other branch of the spectral curve, passing through the same critical point and lying in the same plane, is tangent to the straight line

$$\Omega=\Omega_0+\alpha^-(\lambda-\lambda_0), \quad (12)_2$$

where α^- is the slope of the line given by eqn $(10)_2$.

Let ϕ be the angle between the two tangent lines with slopes α^+ and α^-. Then from eqn (10)

$$\frac{1}{2}\tan\phi=-\frac{(F^2_{\Omega_0\lambda_0}-F_{\Omega_0\Omega_0}F_{\lambda_0\lambda_0})^{1/2}}{(F_{\Omega_0\Omega_0}+F_{\lambda_0\lambda_0})}, \quad (13)$$

and the angle ϕ between the two branches of the curve is real when the numerator is real. In the particular case when $F_{\Omega_0\lambda_0} \equiv 0$ and $F_{\Omega_0\Omega_0} < 0$, this equation takes the simple form

$$\frac{1}{2}\tan\phi = \left[\left(-\frac{F_{\Omega_0\Omega_0}}{F_{\lambda_0\lambda_0}}\right)^{1/2} - \left(-\frac{F_{\lambda_0\lambda_0}}{F_{\Omega_0\Omega_0}}\right)^{1/2}\right]^{-1}. \quad (14)\dagger$$

Similarily if we consider a plane $\lambda = \lambda_0$ passing through the spectral surface of the cone, the equations of the tangent lines to the curve of intersection at the double point P_0 where $F_{\Omega_0\Omega_0} \neq 0$, are given by

$$\Omega = \Omega_0 + \beta^{\pm}(\kappa - \kappa_0), \quad (15)$$

where

$$\begin{matrix}\beta^+\\ \beta^-\end{matrix} = [-F_{\Omega_0\kappa_0} \pm (F^2_{\Omega_0\kappa_0} - F_{\Omega_0\Omega_0}F_{\kappa_0\kappa_0})^{1/2}]F^{-1}_{\Omega_0\Omega_0}. \quad (16)$$

When the discriminant is real, the two values of β are distinct and real and the double point is like the node of a lemniscate. The angle between the two real distinct tangent lines is given by

$$\frac{1}{2}\tan\phi = -\frac{(F^2_{\Omega_0\kappa_0} - F_{\Omega_0\Omega_0}F_{\kappa_0\kappa_0})^{1/2}}{(F_{\Omega_0\Omega_0} + F_{\kappa_0\kappa_0})}. \quad (17)$$

For certain values of the parameters, the discriminant in eqns (10) and (16) can be zero. In this case we obtain two coincident tangents and the corresponding angle ϕ is zero. In such a situation the two branches of the curve touch each other, and there is in general a cusp at the contact point P_0. Whether the cusp is of the *Keratoid* (i.e. like a *horn*) or of the *Ramphoid* (i.e. like a *beak*) species requires further examination of the higher order terms.

Consider now a plane $\Omega = \Omega_0$, parallel to the (λ, κ)-plane at the point P_0 and passing through the spectral surface. We tentatively assume that $F_{\kappa_0\kappa_0} \neq 0$. Then the equations of the tangent lines to the level-curves of constant frequency and passing through the critical point P_0 are given by

$$\kappa = \kappa_0 + \gamma^{\pm}(\lambda - \lambda_0), \quad (18)$$

where

$$\begin{matrix}\gamma^+\\ \gamma^-\end{matrix} = [-F_{\lambda_0\kappa_0} \pm (F^2_{\lambda_0\kappa_0} - F_{\lambda_0\lambda_0}F_{\kappa_0\kappa_0})^{1/2}]F^{-1}_{\kappa_0\kappa_0}. \quad (19)$$

When the descriminant is real, the two tangent lines are real and subtend an angle

$$\frac{1}{2}\tan\phi = -\frac{(F^2_{\lambda_0\kappa_0} - F_{\lambda_0\lambda_0}F_{\kappa_0\kappa_0})^{1/2}}{(F_{\lambda_0\lambda_0} + F_{\kappa_0\kappa_0})}. \quad (20)$$

When $F_{\lambda_0\kappa_0} \equiv 0$, $F_{\lambda_0\lambda_0} > 0$ and $F_{\kappa_0\kappa_0} < 0$, this takes the simple form

$$\frac{1}{2}\tan\phi = \left[\left(-\frac{F_{\kappa_0\kappa_0}}{F_{\lambda_0\lambda_0}}\right)^{1/2} - \left(-\frac{F_{\lambda_0\lambda_0}}{F_{\kappa_0\kappa_0}}\right)^{1/2}\right]^{-1}. \quad (21)$$

When the two tangent lines are real and distinct, the contact point is a node, as discussed earlier.

We now tentatively assume that in the case of Floquet waves the two mixed partial derivatives $F_{\Omega_0\kappa_0}$ and $F_{\lambda_0\kappa_0}$ both vanish simultaneously at the critical point. In this case, the equation of the quadratic surface takes the form

$$(\Omega - \Omega_0)^2F_{\Omega_0\Omega_0} + (\lambda - \lambda_0)^2F_{\lambda_0\lambda_0} + (\kappa - \kappa_0)^2F_{\kappa_0\kappa_0} + 2(\Omega - \Omega_0)(\kappa - \kappa_0)F_{\Omega_0\kappa_0} = 0. \quad (22)$$

†We have here used the fact that $F_{..}/(-F_{..})^{1/2} = -(|F_{..}|)^{1/2}$ when $F_{..} < 0$.

Consider now an affine-mapping

$$\bar{\lambda} = \lambda,$$
$$\bar{\kappa} = \bar{\kappa}_0 + (\kappa - \kappa_0)\Delta F^{-1}_{\Omega_0\Omega_0},$$
$$\bar{\Omega} = \bar{\Omega}_0 + (\Omega - \Omega_0) + (\kappa - \kappa_0) F_{\Omega_0\kappa_0} F^{-1}_{\Omega_0\Omega_0}, \tag{23}$$

where $\Delta \equiv (F^2_{\Omega_0\kappa_0} - F_{\Omega_0\Omega_0}F_{\kappa_0\kappa_0})^{1/2}$. This mapping is (1 : 1) when $\Delta \neq 0$. Under this affine-mapping the quadric surface takes the canonical form

$$\frac{(\bar{\lambda} - \bar{\lambda}_0)^2}{-F_{\Omega_0\Omega_0}} + \frac{(\bar{\kappa} - \bar{\kappa}_0)^2}{F_{\lambda_0\lambda_0}} = \frac{(\bar{\Omega} - \bar{\Omega}_0)^2}{F_{\lambda_0\lambda_0}}. \tag{24}$$

When $F_{\Omega_0\Omega_0} < 0$, this equation represents an elliptic cone in $(\bar{\Omega}, \bar{\lambda}, \bar{\kappa})$-coordinate space, symmetric with respect to all three coordinate planes passing through the point $\bar{P}_0$: $\{\bar{\Omega}_0, \bar{\lambda}_0, \bar{\kappa}_0\}$. In fact, the generator of the conical surface is a straight line passing through the points $\bar{P}_0$ and $\bar{Q}_0$, where the point $\bar{Q}_0$ lies on the ellipse

$$\frac{(\bar{\lambda} - \bar{\lambda}_0)^2}{-F_{\Omega_0\Omega_0}} + \frac{(\bar{\kappa} - \bar{\kappa}_0)^2}{F_{\lambda_0\lambda_0}} = 1, \quad \bar{\Omega} - \bar{\Omega}_0 = C, \tag{25}$$

with semiaxes $(-F_{\Omega_0\Omega_0})^{1/2}$ and $(F_{\lambda_0\lambda_0})^{1/2}$, respectively.

From the canonical representation of the cone with $F_{\Omega_0\Omega_0} < 0$, we immediately see that a plane

$$(\Omega - \Omega_0)F_{\Omega_0\Omega_0} + (\kappa - \kappa_0)F_{\Omega_0\kappa_0} = 0, \tag{26}$$

cuts the quadric surface only at one isolated point (λ_0, κ_0) when $(F_{\Omega_0\Omega_0}F_{\lambda_0\lambda_0}) < 0$ and Δ is real. The direction-cosines of the normal to the plane are given by

$$\cos\theta = \frac{F_{\Omega_0\Omega_0}}{(F^2_{\Omega_0\Omega_0} + F^2_{\Omega_0\kappa_0})^{1/2}}, \quad \sin\theta = \frac{F_{\Omega_0\kappa_0}}{(F^2_{\Omega_0\Omega_0} + F^2_{\Omega_0\kappa_0})^{1/2}}, \tag{27}$$

where θ is the angle which the normal to the plane makes with the positive κ-axis.

3. RAYLEIGH–LAMB WAVES

As a simple application of the preceding analysis we consider the problem of coalescence of cut-off frequencies for Rayleigh–Lamb waves in an infinite, isotropic, elastic plate of thickness $2b$, with the two major faces of the plate free of traction[1]. For motion symmetric with respect to the middle plane of the plate, the frequency equation is

$$F(\Omega, \kappa) \equiv \kappa^2\alpha\beta \sin\alpha \cos\beta + \left(\kappa^2 - \frac{\pi^2}{8}\Omega^2\right)^2 \sin\beta \cos\alpha = 0, \tag{28}$$

where

$$\alpha = \sqrt{\left[\left(\frac{\pi\Omega}{2c}\right)^2 - \kappa^2\right]}, \qquad \beta = \sqrt{\left[\left(\frac{1}{2}\pi\Omega\right)^2 - \kappa^2\right]},$$
$$c = v_L/v_S, \qquad \Omega = 2\omega b/(\pi v_S)$$
$$v_L^2 = (\lambda + 2\mu)/\rho, \qquad v_S^2 = \mu/\rho, \qquad \kappa = b\xi. \tag{29}$$

In this equation b is the half-thickness of the plate, λ and μ are the two Lamé's constants of linear elasticity, ρ is the mass density of the plate material, c is the ratio of the dilatational and shear wave speeds in an unbounded, elastic, isotropic medium, ω is the circular frequency in radians per unit of time, Ω is the non-dimensional frequency, α and β are the non-dimensional wave-numbers in the thickness direction and κ is the non-dimensional wave-number in the direction of wave propagation.

From eqn (28) we find that the first derivative F_κ vanishes when $\kappa = 0$, and the function F along with the first derivative F_Ω vanish simultaneously for $\kappa = 0$, when

$$\sin\beta = 0, \qquad \cos\alpha = 0, \tag{30}$$

that is, when

$$\begin{aligned} \Omega &= q, \qquad q = 2, 4, 6, \ldots \\ \Omega &= cp, \qquad p = 1, 3, 5, \ldots \end{aligned} \tag{31}$$

Thus the coordinates of the critical point are $\kappa_0 = 0$ and $\Omega_0 = q = cp$. It therefore follows that there will be coalescence of cut-off frequencies at $\kappa_0 = 0$, when the two Lamé's constants satisfy the equality $c = q/p$, that is when $\lambda/\mu = (q/p)^2 - 2$, or in terms of Poisson's ratio $\nu = \frac{1}{2}[(q/p)^2 - 2]/[(q/p)^2 - 1]$.

At the critical point $F_{\Omega_0\kappa_0} \equiv 0$, and the remaining two second derivatives are

$$\begin{aligned} F_{\Omega_0\Omega_0} &= -\frac{\pi^6\Omega_0^4}{128c}(\cos\beta\,\sin\alpha)_0, \\ F_{\kappa_0\kappa_0} &= \frac{\pi^2\Omega_0^2}{2c}(\cos\beta\,\sin\alpha)_0. \end{aligned} \tag{32}$$

Therefore, it follows from eqns (15) and (32) that the slopes of the two tangent lines to the spectral curves at the point of coalescence are

$$\left(\frac{\partial\Omega}{\partial\kappa}\right)^{\pm} = \mp(-F_{\kappa_0\kappa_0}/F_{\Omega_0\Omega_0})^{1/2} = \mp\frac{8}{\pi^2 q}. \tag{33}$$

These are the slopes of the thickness-shear and thickness-stretch (reflected about $\kappa = 0$) branches at coalescence of cut-off frequencies and agree with the results first obtained by Mindlin[1, p. 2.44]. The equations of the tangent lines to the spectral curves are

$$\Omega^{\pm} = q \mp \frac{8\kappa}{\pi^3 q}, \tag{34}$$

and the angle between the two tangent lines in the (Ω, κ)-plane is

$$\tan\phi = 16q/[(8/\pi)^2 - (\pi q)^2]. \tag{35}$$

The point of coalescence of the two spectral branches at zero wave-number is a contact point of order one. Therefore the two spectral curves meeting at this point do not cross each other, but rather intersect at two coincident points.

Finally it may be mentioned that there may be other critical points in the spectrum, possibly for imaginary values of the wave-number κ. Simultaneous solution of the three equations $F = F_\Omega = F_\kappa = 0$ determines Ω, κ and Poisson's ratio ν for which such critical points may exist.

4. FLOQUET WAVES OF THE *SH*-TYPE

As an example of critical points of a spectral surface, we consider the propagation of time-harmonic, horizontally polorized shear waves in a periodically layered medium of infinite extent. The material medium is assumed to consist of periodically repeating, perfectly bonded layers. Two such contiguous layers of thickness $\{2h; 2h'\}$ form a typical unit cell, with lattice distance $d \equiv 2(h + h')$. The two elastic constants in each layer of the unit cell are $\{h; h'\}$: $\{(\lambda, \mu); (\lambda', \mu')\}$ and the mass density of the layers of a typical unit cell is $\{h; h'\}:\{\rho; \rho'\}$, respectively. In a recent paper by Delph, Herrmann and Kaul[3], it has been shown that the spectral equation in this case may be obtained by making use of Floquet's theory of differential equations with periodic coefficients[4]. In non-dimensional form the spectral equation is

$$F(\Omega, \alpha, \beta) \equiv 2\gamma\alpha\beta\mathcal{F}_1(\alpha, \beta, \lambda) + (\gamma^2\alpha^2 + \beta^2)\mathcal{F}_2(\alpha, \beta) = 0, \tag{36}$$

where

$$\begin{aligned}
\mathcal{F}_1(\alpha, \beta, \lambda) &= [\cos \pi(1+t)\lambda - \cos \pi\alpha \cos \pi t\beta],\\
\mathcal{F}_2(\alpha, \beta) &= \sin \pi\alpha \sin \pi t\beta,\\
\alpha &= (\Omega^2 - \kappa^2)^{1/2},\\
\beta &= (\sigma^2\Omega^2 - \kappa^2)^{1/2},\\
\gamma &= \mu/\mu'; \quad t = h'/h; \quad \sigma^2 = (c/c')^2,\\
c^2 &= \mu/\rho; \quad (c')^2 = \mu'/\rho',\\
\lambda &= (2h/\pi)k_2; \quad \kappa = (2h/\pi)k_3,\\
\Omega &= \omega/\omega_s; \quad \omega_s = (\pi/2h)(\mu/\rho)^{1/2}.
\end{aligned} \tag{37}$$

In these equations λ and κ are the two non-dimensional wave-numbers, λ being the wave-number in the direction of the periodicity of the layering, and κ being the wave-number in the mutually orthogonal non-periodic direction. The reference frequency ω_s is the lowest antisymmetric thickness-shear frequency of an infinite, homogeneous, isotropic plate of thickness $2h$, and Ω is the non-dimensional frequency.

The spectral eqn (36) represents a surface in three-dimensional $(\Omega, \lambda, \kappa)$-affine space, and according to Floquet theory the spectrum is periodic in the direction of the layering with real period $\lambda = 2/(1+t)$. Therefore in the direction of the layering the width of the Brillouin zone is $\lambda_0 = 1/(1+t)$. Due to inversion symmetry and the periodic property of the spectral function F, the slopes of the dispersion surface $(\partial\Omega/\partial\lambda)_\kappa$ and $(\partial\kappa/\partial\lambda)_\Omega$ at the end-points of the Brillouin zone are zero, in general. However, at certain singular points of the dispersion surface, these derivatives may acquire a non-zero value, which suggests that the slope of the surface at these singular points may be discontinuous. This analysis is aimed at investigating the nature of the surface in the immediate neighborhood of these exceptional points.

From the analysis presented in Section 2, the singular points are those roots of frequency eqn (36), where the three first partial derivatives vanish simultaneously. In the present case the three first partial derivatives of the function F are

$$\begin{aligned}
F_\lambda &= -2\pi\alpha\beta\gamma(1+t)\sin \pi(1+t)\lambda,\\
F_\kappa &= -2\kappa\gamma\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)[\cos \pi(1+t)\lambda - \cos \pi\alpha \cos \pi t\beta] - \frac{\pi\kappa}{\beta}[t\gamma^2\alpha^2 + (t+2\gamma)\beta^2]\sin \pi\alpha \cos \pi t\beta\\
&\quad - \frac{\pi\kappa}{\alpha}[\beta^2 + (\gamma+2t)\gamma\alpha^2]\cos \pi\alpha \sin \pi t\beta - 2\kappa(1+\gamma^2)\sin \pi\alpha \sin \pi t\beta,\\
F_\Omega &= 2\gamma\Omega\left(\frac{\beta}{\alpha}+\sigma^2\frac{\alpha}{\beta}\right)[\cos \pi(1+t)\lambda - \cos \pi\alpha \cos \pi t\beta]\\
&\quad + \frac{\pi\Omega}{\beta}[(\gamma\sigma\alpha)^2 t + (2\gamma + t\sigma^2)\beta^2]\sin \pi\alpha \cos \pi t\beta\\
&\quad + \frac{\pi\Omega}{\alpha}[(\gamma + 2t\sigma^2)\gamma\alpha^2 + \beta^2]\cos \pi\alpha \sin \pi t\beta\\
&\quad + 2\Omega(\gamma^2 + \sigma^2)\sin \pi\alpha \sin \pi t\beta.
\end{aligned} \tag{38}$$

Now consider the equation $F_\lambda = 0$, which has the roots

$$\lambda_0 = s/(1+t), \quad s = 0, 1, 2, 3, \ldots \tag{39}$$

provided α and β are non-zero. These roots define the end-points of the Brillouin zone, which indicates that the critical points of the dispersion surface are located at the end-points of the zone. At the end-points of the Brillouin zone, eqn (36) can be factored into the product of two simple transcendental equations. Thus, when s is an *even* integer, eqn (36) uncouples into two equations

$$\begin{aligned} &\gamma\alpha \sin\frac{\pi}{2}\alpha \cos\frac{\pi}{2}t\beta + \beta \cos\frac{\pi}{2}\alpha \sin\frac{\pi}{2}t\beta = 0,\\ &\gamma\alpha \cos\frac{\pi}{2}\alpha \sin\frac{\pi}{2}t\beta + \beta \sin\frac{\pi}{2}\alpha \cos\frac{\pi}{2}t\beta = 0, \end{aligned} \tag{40$_1$}$$

and when s is an *odd* integer, the two uncoupled equations are

$$\begin{aligned} &\gamma\alpha \sin\frac{\pi}{2}\alpha \sin\frac{\pi}{2}t\beta - \beta \cos\frac{\pi}{2}\alpha \cos\frac{\pi}{2}t\beta = 0,\\ &\gamma\alpha \cos\frac{\pi}{2}\alpha \cos\frac{\pi}{2}t\beta - \beta \sin\frac{\pi}{2}\alpha \sin\frac{\pi}{2}t\beta = 0. \end{aligned} \tag{40$_2$}$$

In the $(\Omega, \kappa)_{\lambda_0}$-plane, the locus of the roots of these equations represent plane curves and we are interested in the coordinates of those points where the plane curves have a contact point of order 1. In order that two plane curves have a contact point of order 1, it is necessary and sufficient that at the point of contact, the functions along with their first partial derivatives be equal. Thus, consider eqn $(40)_1$, which is valid for even values of integer s. These two equations have a zero-order contact when the roots of the two equations are the same. This requires that for even values of integer s the roots must satisfy one of the following three conditions

(i)

$$\sin\frac{\pi}{2}\alpha = 0, \quad \alpha_0 = 2n$$

$$\sin\frac{\pi}{2}t\beta = 0, \quad t\beta_0 = 2m \tag{41$_1$}$$

(ii)

$$\cos\frac{\pi}{2}\alpha = 0, \quad \alpha_0 = (2n-1)$$

$$\cos\frac{\pi}{2}t\beta = 0, \quad t\beta_0 = (2m-1) \tag{41$_2$}$$

(iii)

$$\tan\frac{\pi}{2}\alpha + \tan\frac{\pi}{2}t\beta = 0, \quad \alpha_0 = 2n/(1+\gamma t)$$

$$\gamma\alpha - \beta = 0, \quad \beta_0 = 2n\gamma/(1+\gamma t) \tag{41$_3$}$$

where in these equations $n, m = 1, 2, 3, \ldots$. Similarly for odd values of integer s, the roots of eqn $(40)_2$ are contact points of order zero, if they satisfy one of the following three conditions

(i)

$$\sin\frac{\pi}{2}\alpha = 0, \quad \alpha_0 = 2n$$

$$\cos\frac{\pi}{2}t\beta = 0, \quad t\beta_0 = (2m-1) \tag{42$_1$}$$

(ii)

$$\cos\frac{\pi}{2}\alpha = 0, \quad \alpha_0 = (2n-1)$$

$$\sin\frac{\pi}{2}t\beta = 0, \quad \beta_0 = 2m \tag{42$_2$}$$

(iii)

$$\tan\frac{\pi}{2}\alpha - \cot\frac{\pi}{2}t\beta = 0, \quad \alpha_0 = (2n-1)/(1+\gamma t)$$

$$\gamma\alpha - \beta = 0, \quad \beta_0 = \gamma(2n-1)/(1+\gamma t) \tag{42$_3$}$$

where in these equations $n, m = 1, 2, 3, \ldots$. We may here remark that from the zero-order contact points we have specifically excluded the roots $\alpha_0 = 0$, $\beta_0 = 0$. It has been shown in Ref. [3] that $\alpha_0 = 0$ cannot be a root of eqn (36) and thus is not a possible contact point. We now show that $\beta_0 = 0$ is also an inadmissible contact point.

From the first of eqn $(40)_1$, we find that when $\beta_0 = 0$, the equation is satisfied when $\alpha \sin(\pi/2)\alpha = 0$. Excluding the root $\alpha_0 = 0$, the non-trivial roots of this equation are $\alpha_0 = 2n$, $n = 1, 2, 3, \ldots$. Now consider the second of eqn $(40)_1$. This equation is satisfied identically when $\beta_0 = 0$. Therefore, using the turning point theorem [5], we get the limiting form of the frequency equation

$$\pi\alpha\gamma t \cos\frac{\pi}{2}\alpha + 2\sin\frac{\pi}{2}\alpha = 0, \tag{43}$$

whose roots give us the admissible values of α when $\beta_0 = 0$. Since $\alpha_0 = 2n$ fails to satisfy this transcendental frequency equation, it can be concluded that the two eqns $(40)_1$ do not have a contact point when $\beta_0 = 0$. A similar analysis shows that the two eqns $(40)_2$ have also no contact points when $\beta_0 = 0$.

At these zero-order contact points, the three first derivatives F_{κ_0}, F_{Ω_0} and F_{λ_0} also vanish simultaneously. Hence, the critical points of the spectral surface are contact points of order one, with coordinates

$$\begin{aligned} \lambda_0 &= s/(1+t), \quad s = 0, 1, 2, 3, \ldots \\ \Omega_0 &= [(\alpha_0^2 - \beta_0^2)/(1-\sigma^2)]^{1/2}, \\ \kappa_0 &= [(\alpha_0^2\sigma^2 - \beta_0^2)/(1-\sigma^2)]^{1/2}, \end{aligned} \tag{44}$$

where for s-even (odd) the values of α_0 and β_0 are given by eqn (41) (eqn 42). According to the discussion in Section 2, these are the coordinates of the conical points where the spectral surfaces touch each other at two coincident points without crossing each other.

The mode shapes corresponding to the uncoupled forms of the frequency eqns $(40)_{1,2}$ have been fully discussed in Ref. [3]. Avoiding *minutiae* we may briefly reiterate that the first of eqn $(40)_1$ is valid when the motion with respect to the respective mid-planes of each of the two laminae comprising the unit cell is symmetric, and the second of eqn $(40)_1$ holds when the motion of each lamina in the cell is antisymmetric. Similarly, the first of eqn $(40)_2$ is valid when in the portion of the cell with unprimed constants the motion is symmetric, and in the remaining portion of the cell with primed constants, the motion is antisymmetric. In both cases the symmetry is defined with respect to the mid-planes of the layers. The converse is true in the case of second of eqn $(40)_2$. Thus, the conical points with coordinates given by eqns (44) and (41), corresponding to the end-points of the Brillouin zones (s-even), are first order contact points of the dispersion spectra belonging to the symmetric-symmetric and antisymmetric-antisymmetric families. Similarly when s is an odd integer, the conical points with coordinates given by eqns (44) and (42), are first order contact points of the dispersion spectra belonging to the symmetric-antisymmetric and antisymmetric-symmetric families.

To investigate the nature of the dispersion surface in the neighborhood of the conical points, we need the six second derivatives of the spectral function $F(\Omega, \alpha, \beta)$. On differentiating again eqn (38), we get

$$F_{\lambda\lambda} = -2\pi^2\alpha\beta\gamma(1+t)^2\cos\pi(1+t)\lambda, \tag{45$_1$}$$

$$F_{\Omega\lambda} = -2\pi\gamma(1+t)\Omega\left(\frac{\beta}{\alpha} + \sigma^2\frac{\alpha}{\beta}\right)\sin\pi(1+t)\lambda, \tag{45$_2$}$$

$$F_{\kappa\lambda} = 2\pi\gamma(1+t)\kappa\left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right)\sin\pi(1+t)\lambda, \tag{45$_3$}$$

$$F_{\Omega\Omega} - \frac{1}{\Omega}F_{\Omega} = -2\gamma\Omega^2\frac{(\sigma^2\alpha^2 - \beta^2)^2}{(\alpha\beta)^3}\cos\pi(1+t)\lambda$$

$$+2\Omega^2\left[\frac{\gamma(\sigma^2\alpha^2-\beta^2)^2}{(\alpha\beta)^3}+\pi^2(\gamma+t\sigma^2)\left(\frac{\beta}{\alpha}+t\gamma\sigma^2\frac{\alpha}{\beta}\right)\right]\cos\pi\alpha\cos\pi t\beta$$

$$-\pi^2\Omega^2\left[\left(\frac{\beta}{\alpha}+t\gamma\sigma^2\frac{\alpha}{\beta}\right)^2+(\gamma+t\sigma^2)^2\right]\sin\pi\alpha\sin\pi t\beta$$

$$+\pi\Omega^2\left[2\gamma\frac{\beta^2}{\alpha^2}-t\gamma^2\sigma^4\frac{\alpha^2}{\beta^2}+\sigma^2(4\gamma+4t\gamma^2+3t\sigma^2)\right]\frac{1}{\beta}\sin\pi\alpha\cos\pi t\beta$$

$$+\pi\Omega^2\left[2t\gamma\sigma^4\frac{\alpha^2}{\beta^2}-\frac{\beta^2}{\alpha^2}+(4\sigma^2+4t\gamma\sigma^2+3\gamma^2)\right]\frac{1}{\alpha}\cos\pi\alpha\sin\pi t\beta, \qquad (45)_4$$

$$F_{\kappa\kappa}-\frac{1}{\kappa}F_\kappa=-2\gamma\kappa^2\frac{(\alpha^2-\beta^2)^2}{(\alpha\beta)^3}\cos\pi(1+t)\lambda$$

$$+2\kappa^2\left[\gamma\frac{(\alpha^2-\beta^2)^2}{(\alpha\beta)^3}+\pi^2(\gamma+t)\left(\frac{\beta}{\alpha}+\gamma t\frac{\alpha}{\beta}\right)\right]\cos\pi\alpha\cos\pi t\beta$$

$$-\pi^2\kappa^2\left[(\gamma+t)^2+\left(\frac{\beta}{\alpha}+\gamma t\frac{\alpha}{\beta}\right)^2\right]\sin\pi\alpha\sin\pi t\beta$$

$$+\pi\kappa^2\left[(3t+4\gamma+4t\gamma^2)+\gamma\left(2\frac{\beta^2}{\alpha^2}-t\gamma\frac{\alpha^2}{\beta^2}\right)\right]\frac{1}{\beta}\sin\pi\alpha\cos\pi t\beta$$

$$+\pi\kappa^2\left[(4+4\gamma t+3\gamma^2)+\left(2t\gamma\frac{\alpha^2}{\beta^2}-\frac{\beta^2}{\alpha^2}\right)\right]\frac{1}{\alpha}\cos\pi\alpha\sin\pi t\beta, \qquad (45)_5$$

$$F_{\Omega\kappa}=2\gamma\Omega\kappa\frac{(\alpha^2-\beta^2)(\sigma^2\alpha^2-\beta^2)}{(\alpha\beta)^3}\cos\pi(1+t)\lambda$$

$$-\Omega\kappa\left[2\gamma\frac{(\alpha^2-\beta^2)(\sigma^2\alpha^2-\beta^2)}{(\alpha\beta)^3}+\pi^2\left\{[2\gamma+(1+\sigma^2)t]\frac{\beta}{\alpha}+t\gamma[2t\sigma^2+(1+\sigma^2)\gamma]\frac{\alpha}{\beta}\right\}\right]$$

$$\times\cos\pi\alpha\cos\pi t\beta+\pi^2\kappa\Omega\left[2\gamma t(1+\sigma^2)+(\gamma-\sigma t)^2+\left(t\gamma\sigma\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)^2\right]\sin\pi\alpha\sin\pi t\beta$$

$$-\pi\kappa\Omega\left[2\gamma\left\{(1+\gamma t)(1+\sigma^2)+\left(\frac{\beta}{\alpha}\right)^2\right\}+t\sigma^2\left(3-\gamma^2\frac{\alpha^2}{\beta^2}\right)\right]\frac{1}{\beta}\sin\pi\alpha\cos\pi t\beta$$

$$-\pi\kappa\Omega\left[2(1+\gamma t)(1+\sigma^2)+3\gamma^2+2\gamma t\sigma^2\frac{\alpha^2}{\beta^2}-\frac{\beta^2}{\alpha^2}\right]\frac{1}{\alpha}\cos\pi\alpha\sin\pi t\beta. \qquad (45)_6$$

For the purpose of illustration, consider now a typical conical point located at the end of the Brillouin zone $s=3$, with abscissa

$$\lambda_0=3/(1+t), \qquad (46)_1$$

and in accordance with eqn $(42)_1$, for odd values of integer s, let

$$\alpha_0=2n, \qquad n=1$$
$$t\beta_0=(2m-1), \qquad m=1. \qquad (46)_2$$

For this particular choice of the conical point, henceforth labeled P_0^*, the coordinates from eqn (44) are

$$\lambda_0=3/(1+t),$$
$$\Omega_0=[\{4n^2-(2m-1)^2/t^2\}/(1-\sigma^2)]^{1/2},$$
$$\kappa_0=[\{4\sigma^2n^2-(2m-1)^2/t^2\}/(1-\sigma^2)]^{1/2}. \qquad (47)$$

At this point P_0^*, the two mixed derivatives $F_{\Omega_0\lambda_0}$ and $F_{\kappa_0\lambda_0}$ vanish simultaneously, and the remaining four second derivatives are explicitly given by

$$F_{\lambda_0\lambda_0} = 2\pi^2\gamma(1+t)^2(2n)(2m-1)/t,$$

$$F_{\Omega_0\Omega_0} = -2\pi^2\Omega_0^2(\gamma+t\sigma^2)\frac{[(2m-1)^2+\gamma\sigma^2t^3(2n)^2]}{2n(2m-1)t},$$

$$F_{\kappa_0\kappa_0} = -2\pi^2\kappa_0^2(\gamma+t)\frac{[(2m-1)^2+\gamma t^3(2n)^2]}{2n(2m-1)t},$$

$$F_{\Omega_0\kappa_0} = \frac{\pi^2\Omega_0\kappa_0}{2n(2m-1)t}[\{2\gamma+(1+\sigma^2)t\}(2m-1)^2+\{2t\sigma^2+\gamma(1+\sigma^2)\}(2n)^2\gamma t^3]. \quad (48)$$

Therefore the locus of the tangents to all curves on the spectral surface passing through the point P_0^* is given by

$$(\Omega-\Omega_0)^2F_{\Omega_0\Omega_0}+(\lambda-\lambda_0)^2F_{\lambda_0\lambda_0}+(\kappa-\kappa_0)^2F_{\kappa_0\kappa_0}+2(\Omega-\Omega_0)(\kappa-\kappa_0)F_{\Omega_0\kappa_0}=0, \quad (49)$$

where the coordinates of the point P_0^*: $(\Omega_0, \lambda_0, \kappa_0)$ are given by eqn (47) and the four second derivatives are explicitly given by eqn (48). Now

$$\Delta = \pm\frac{\pi^2\Omega_0\kappa_0}{2n(2m-1)}(1-\sigma^2)[(2m-1)^2-(2n\gamma t)^2], \quad (50)$$

which is the discriminant of the quadratic form (49), and is non-zero when

$$\sigma \neq 1, \quad (2m-1)/2n \neq \gamma t. \quad (51)$$

Therefore there exists a non-singular affine-mapping (23), which reduces the quadric surface (49) to an elliptic cone (24), since $F_{\Omega_0\Omega_0}<0$. The eccentricity of the ellipse is $e=(1+F_{\Omega_0\Omega_0}/F_{\lambda_0\lambda_0})^{1/2}$, where

$$F_{\Omega_0\Omega_0}/F_{\lambda_0\lambda_0} = -\frac{\Omega_0^2(\gamma+t\sigma^2)}{\gamma(1+t)^2}\left[\frac{1}{(2n)^2}+\frac{\gamma\sigma^2t^3}{(2m-1)^2}\right]. \quad (52)$$

Using the extended zone scheme, the shape of the spectral surface in the neighborhood of the conical point P_0^* is shown qualitatively in Fig. 1.

We now consider plane sections of the surface, passing through the critical point P_0^* and parallel to the coordinate planes. These sections are plane curves and we write down explicitly the equations of the tangent lines to the sectional curves at the point P_0^*, and the angle between them.

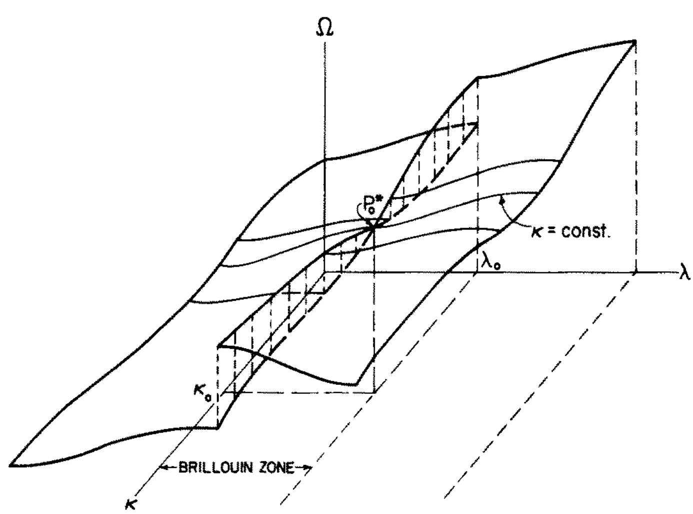

Fig. 1. Qualitative sketch of the spectral surface in the neighborhood of a typical conical point P_0^*.

(i) *Plane* $\kappa = \kappa_0$ *passing through the conical point* P_0^*

In this case the equations of the tangent lines are given by eqn (7). When $F_{\Omega_0\lambda_0} \equiv 0$ and $F_{\Omega_0\Omega_0} < 0$, the slopes of the two tangent lines are

$$\alpha^{\pm} = \mp(-F_{\lambda_0\lambda_0}/F_{\Omega_0\Omega_0})^{1/2}. \tag{53}$$

Substituting the values of second derivatives from (48), we get

$$\alpha^{\pm} = \mp\frac{(1+t)(2n)(2m-1)}{\Omega_0[(1+(\sigma^2/\gamma)t)\{(2m-1)^2+\gamma\sigma^2t^3(2n)^2\}]^{1/2}}. \tag{54}$$

For $n = 1$, $m = 1$, $\gamma = 0.02$, $\sigma^2 = 0.06$ and $t = 4.0$, the coordinates of the point P_0^* are $\Omega_0 = 2.04666$, $\lambda_0 = 0.600$ and $\kappa_0 = 0.43455$. From eqn (54) the slope of the tangent lines to the spectral curves passing through the point P_0^* are $\alpha^{\pm} = \mp 1.18525$. The tangent of the angle between the two tangent lines is

$$\tan\phi = \frac{-2\Omega_0(1+t)\left(1+\frac{\sigma^2}{\gamma}t\right)^{1/2}[(2m-1)^2+\gamma\sigma^2t^3(2n)^2]^{1/2}}{\left\{(2m-1)(2n)(1+t)^2-\Omega_0^2\left(1+\frac{\sigma^2}{\gamma}t\right)\left[\left(\frac{2m-1}{2n}\right)+\gamma\sigma^2t^3\left(\frac{2n}{2m-1}\right)\right]\right\}}. \tag{55}$$

For the given values of the parameters, $\tan\phi = -5.855593$, or equivalently $\phi = 99°.41'.28''(80°.18'.32'')$.

The tangent line with slope α^- and passing through the point P_0^*, is shown in Fig. 2(c), which is plotted on an extended zone scheme. At the point of coalescence, the two curves defined by the intersection of the plane $\kappa = \kappa_0$ with the spectral surface in the adjoining Brillouin zones meet at one point with non-zero slope. Consequently, the group velocity in the λ-direction is non-zero at the point of confluence of two spectral curves. In addition, the stopping band disappears completely. The gradual decrease in the width of the stopping band, till it disappears, and its reappearance with gradual increase of the wave-number κ, is exemplified in the sequence of graphs (a) to (e) of Fig. 2.

(ii) *Plane* $\lambda = \lambda_0$ *passing through the conical point* P_0^*

In this case the equations of the tangent lines are given by eqn (15), where $\beta^{\pm}$ are the slopes of the tangent lines passing through the point P_0^*. Substituting the values of the second derivatives $F_{\Omega_0\Omega_0}$, $F_{\kappa_0\kappa_0}$ and $F_{\Omega_0\kappa_0}$ from (48) in eqn (16), we get

$$\beta^+ = \frac{\kappa_0[(2m-1)^2+\gamma t^3(2n)^2]}{\Omega_0[(2m-1)^2+\gamma\sigma^2t^3(2n)^2]},$$

$$\beta^- = \frac{\kappa_0(\gamma+t)}{\Omega_0(\gamma+t\sigma^2)}. \tag{56}$$

In this case the two distinct values of the slopes are both real and the point P_0^* is like the node of a lemniscate. For the same values of the parameters used in case (i), the numerical values of the slopes of the tangent lines to the spectral curves passing through the point P_0^* are $\beta^+ = 0.99404$ and $\beta^- = 3.28282$. The tangent of the angle between the two tangent lines is

$$\tan\phi = \pm\frac{t\Omega_0\kappa_0(1-\sigma^2)[(2m-1)^2-(2n\gamma t)^2]}{\{\Omega_0^2(\gamma+t\sigma^2)[(2m-1)^2+\gamma\sigma^2t^3(2n)^2]+\kappa_0^2(\gamma+t)[(2m-1)^2+\gamma t^3(2n)^2]\}}. \tag{57}$$

For the given values of the parameters, $\tan\phi = 0.536863$, or equivalently $\phi = 28°.13'.47''$ $(151°.46'.13'')$.

The two tangent lines with slopes β^+ and β^- are shown in Fig. 3(c). The contact point is of order 1 and the two curves intersect each other at two coincident points but do not cross. This becomes immediately obvious if one plots the sequence of curves on the reduced zone scheme as

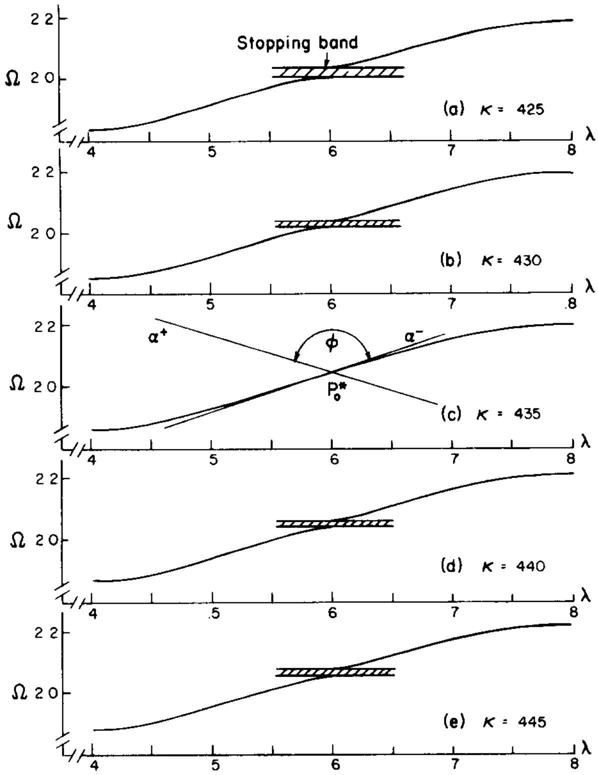

Fig. 2. Behavior of the stopping band due to gradual change in wavenumber κ. Coalition occurs at point P_0^* where group velocity become finite. Stopping band is shown hatched.

shown in Fig. 3. In this figure the spectral curves are plotted on a reduced zone scheme for $\lambda_0 = 3/(1+t) \pm \Delta\lambda$, where (a): $\Delta\lambda = 0.010$, (b): $\Delta\lambda = 0.005$, and (c): $\Delta\lambda = 0$. Figure 3(c) clearly shows how the two spectral curves meet each other at the point P_0^*, without crossing.

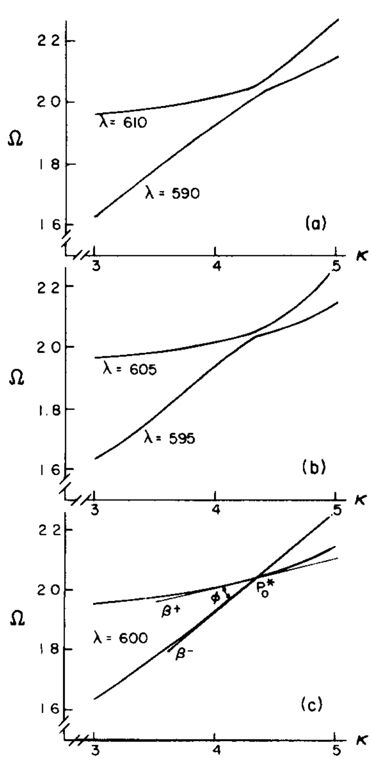

Fig. 3. Behavior of the spectra in the neighborhood of the conical point P_0^*, in the (Ω, κ)-planes for values of $\lambda_0 \pm \Delta\lambda$.

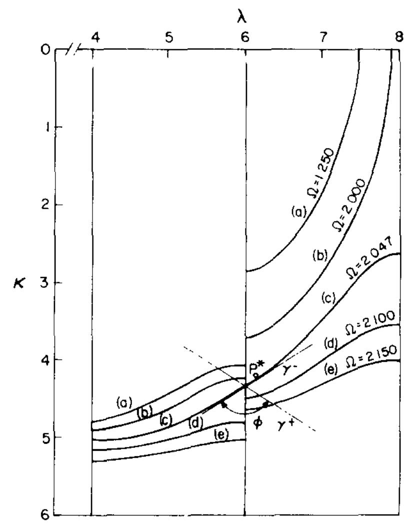

Fig. 4. Isofrequency level-curves in the neighborhood of the conical point P_0^*, in the (λ, κ)-planes for values of $\Omega_0 \pm \Delta\Omega$.

(iii) *Plane $\Omega = \Omega_0$ passing through the conical point* P_0^*

In this case $F_{\kappa_0\kappa_0} \neq 0$, and the equations of the tangent line to the isofrequency level-curves passing through the point P_0^* are given by eqn (18). Since $F_{\lambda_0\kappa_0} \equiv 0$, $F_{\lambda_0\lambda_0} > 0$ and $F_{\kappa_0\kappa_0} < 0$ at P_0^*, the slopes of the two tangent lines are

$$\gamma^{\pm} = \mp(-F_{\lambda_0\lambda_0}/F_{\kappa_0\kappa_0})^{1/2}. \tag{58}$$

Substituting the values of the second derivatives, we get

$$\gamma^{\pm} = \mp\frac{(1+t)(2m-1)(2n)}{\kappa_0[(1+t/\gamma)\{(2m-1)^2+\gamma t^3(2n)^2\}]^{1/2}}. \tag{59}$$

For the same values of the parameters used in (i) and (ii), the numerical values of the slopes of the tangent lines to the isofrequency level-curves are $\gamma^{\pm} = \mp 0.656132$. The tangent of the angle between the two tangent lines passing through the point P_0^* is

$$\tan\phi = \frac{-2\kappa_0(1+t)(2n)(2m-1)[(1+t/\gamma)\{(2m-1)^2+\gamma t^3(2n)^2\}]^{1/2}}{\{\kappa_0^2(1+t/\gamma)[(2m-1)^2+\gamma t^3(2n)^2]-(2n)^2(2m-1)^2(1+t)^2\}}. \tag{60}$$

For the given values of the parameters, $\tan\phi = -2.304213$, or equivalently $\phi = 133°.27'.37''$ $(66°.32'.23'')$. Amongst all planes passing through the point P_0^*, there is one plane which intersects the surface at only one point. The normal to this plane is given by $(\cos\theta, \sin\theta)$: $(-0.42360, 0.90585)$, that is $\theta = 115°.3'.40''$ measured from the positive κ-axis.

To gain some understanding of the spectral surface, a few isofrequency level-curves have been plotted in Fig. 4. Figure 4(c) shows the level-curve passing through the conical point P_0^*.

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